Bankruptcy risk model and empirical tests

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We analyze the size dependence and temporal stability of firm bankruptcy risk in the US economy by applying Zipf scaling techniques. We focus on a single risk factor—the debt-to-asset ratio *R*—in order to study the stability of the Zipf distribution of *R* over time. We find that the Zipf exponent increases during market crashes, implying that firms go bankrupt with larger values of *R*. Based on the Zipf analysis, we employ Bayes's theorem and relate the conditional probability that a bankrupt firm has a ratio *R* with the conditional probability of bankruptcy for a firm with a given *R* value. For 2,737 bankrupt firms, we demonstrate size dependence in assets change during the bankruptcy proceedings. Prepetition firm assets and petition firm assets follow Zipf distributions but with different exponents, meaning that firms with smaller assets adjust their assets more than firms with larger assets during the bankruptcy process. We compare bankrupt firms with nonbankrupt firms by analyzing the assets and liabilities of two large subsets of the US economy: 2,545 Nasdaq members and 1,680 New York Stock Exchange (NYSE) members. We find that both assets and liabilities follow a Pareto distribution. The finding is not a trivial consequence of the Zipf scaling relationship of firm size quantified by employees—although the market capitalization of Nasdaq stocks follows a Pareto distribution, the same distribution does not describe NYSE stocks. We propose a coupled Simon model that simultaneously evolves both assets and debt with the possibility of bankruptcy, and we also consider the possibility of firm mergers.

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Complex systems are commonly coupled together and therefore should be considered and modeled as interdependent. It is important to study the conditions of interaction which may lead to mutual failure, the indicators of such failure, and the behavior of the indicators in times of crisis. As an indicator of economic failure, default risk is defined as the probability that a borrower cannot meet his or her financial obligations, i.e., cannot make principal and/or interest payments (1, 2). Accordingly, it is important to better understand default risk (1–12) and its relation to firm growth (13–17), and how they behave in times of crisis.

We find that book values of assets and debt of the US companies that filed for bankruptcy in the past 20 y follow a Zipf scaling (power-law) distribution. The same is true for the values of assets and debt of nonbankrupt firms comprising the Nasdaq. We focus our attention on a single risk indicator, the debt-to-asset ratio R, in order to analyze stability of the scaling exponent or establish cross-over regions. In order to capture Pareto and Zipf laws, the literature has typically focused on a single Simon model (13, 14, 16, 17) describing a single dynamic system which does not interact with others. We model the growth of debt and asset values using two dependent (coupled) Simon models with two parameters only, bankruptcy rate and another parameter controlling debt-to-asset ratio. The Zipf law scaling predictions of the coupled Simon model are consistent with our empirical findings.

Data Analyzed

Our dataset consists of medium-size and large US companies that filed for bankruptcy protection in the period 1990–2009. We obtain our data from New Generation Research, Inc., which provides one of the most comprehensive bankruptcy datasets currently available on the web. There is also a bankruptcy dataset available at http://bdp.law.harvard.edu/fellows.cfm, but with smaller firms and no debt data. Our dataset includes data on 2,737 public and private firms. The book value of firm assets in the database ranges from 50 million to almost 700 billion US dollars.

i. For each firm in our sample, we know the prepetition book value of firm assets A_a and the effective date of bankruptcy. From the court petition documents we find the petition book value of firm assets A_b, as well as book value of total debt, D_b. As an example, Lehman Brothers filed a petition on September 15, 2008, listing the debt D_b and assets A_b on May 31, 2008. Thus, A_b, A_a, and D_b quantify the debtor's condition before declaring bankruptcy. We are able to obtain A_b and debt D_b for 462 firms. Note that refs. 5, 6, and 12 studied 53, 105, and 585 bankrupt firms, respectively. There is often a substantial change in the debt and assets of a company in the time period preceding bankruptcy. Hence, for each firm, we calculate the debt-to-assets (leverage ratio)

$$R \equiv D_b/A_b \quad [1]$$

from the total debt D_b and assets A_b estimated simultaneously. Note that economics has a parallel treatment, known as Tobin's Q theory of investment which also focuses on a single factor, Q (18).

In the literature on ratio analysis (4, 6, 8), multiple financial ratios are used for predicting probability of default, such as the ratio of total liabilities to total assets. Adding more factors would likely improve the predictive power of the model, so we consider only one risk factor, namely the debt-to-assets ratio R which captures the level of company indebtedness. We use a single ratio for two reasons: (*i*) to make a model as simple as possible, and (*ii*) to simplify our study regarding whether market crashes and global recessions affect the scaling existing in bankruptcy data. In order to relate the probability of bankruptcy to R, we analyze the scaling relations that quantify the probability distribution of firms that entered into bankruptcy proceedings with particular values of A_b and R. Our analysis includes a very few number of young startup firms, for which the age of the firm also factors into the probability of bankruptcy in addition to R. In 2009, we find that the average lifetime of the 215 bankrupt firms analyzed was 35.8 ± 33.9 y and the minimum lifetime was 3 y.

ii. We analyze market capitalization, assets, and liabilities of 2,545 firms traded on the Nasdaq over the 3 y period from 2006 to 2008. We also analyze assets and liabilities of 1,680 firms traded on the New York Stock Exchange (NYSE) in the period from 2007 to 2009. Also, we analyze market capitalization of NYSE members over the period 2002–2007.

Quantitative Methods

Our analysis is closely related to the literature on firm size (19, 20). Analyzing data from the US Census Bureau, ref. 20 reported

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that firm sizes of the US firms follow a Zipf law: The number of firms larger than size s is $s^{-\zeta}$, where $\zeta \simeq 1$. The Zipf distribution is found for the distribution of city sizes (21) and the distribution of firm sizes (20, 22).

The cumulative distribution is a simple transformation of the Zipf rank–frequency relation, where the observations x_i are ordered according to rank r from largest ($r \equiv 1$) to smallest. For Pareto-distributed variables s with cumulative distribution $P(s > x) \sim x^{-\zeta'}$, the Zipf plot of size s versus rank r exhibits a power-law scaling regime with the scaling exponent ζ, where

$$\zeta = 1/\zeta'. \quad [2]$$

Results of Analysis

Fig. 1*A* shows the Zipf plot for prepetition book value of assets A_a. The data are approximately linear in a log–log plot with the exponent

$$\zeta_a = 1.11 \pm 0.01, \quad [3]$$

obtained using the ordinary least-squares regression method. For the US data on firm size (measured by the number of employees), ref. 20 reported the value $\zeta \approx 1$. Hence, prior to filing for bankruptcy protection, the book value of firm assets for companies that later underwent bankruptcy satisfies a scaling relation similar to that in ref. 20. The firms with a rank larger than ≈500 start to deviate from the Zipf law, a result of finite size effects as found in data on firm size (20).

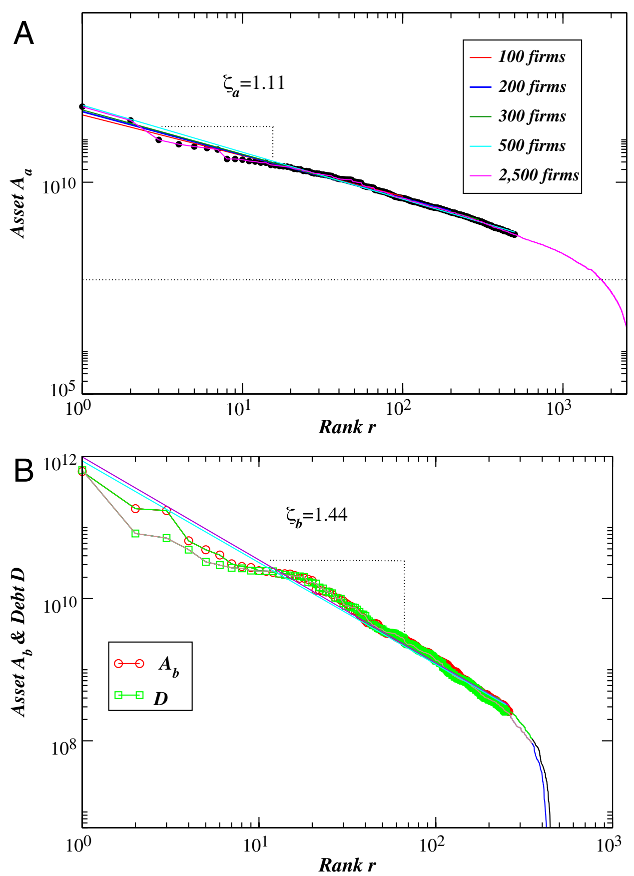

Fig. 1. Zipf plot of US bankrupted firm assets. (*A*) Zipf plot calculated for firms over the last 20 y between prepetition total assets, A_a versus rank. Deviation from the Zipf law is due to the fact that the dataset includes mainly the firms with assets larger than 50 million dollars (dotted line). (*B*) Zipf plot of US bankrupted firms of debt versus rank—462 firms in total—along with Zipf plot of book value asset and rank. The two plots practically overlap.

It is known that the market equity of firms that are close to bankruptcy is typically discounted by traders (10, 12). In order to study if those changes are size dependent during the time of bankruptcy, we test whether there is a difference in scaling behavior between prepetition and petition firm assets. Fig. 1*B* ranks the firm book value of assets A_b and firm debt D_b. We find

$$\zeta_b = \zeta_D = 1.44 \pm 0.01. \quad [4]$$

Note that Fig. 1 includes only firms with the largest values of A_a and A_b. Thus, the firms with the largest bankruptcy adjustments, with potentially small A_b values, are not necessarily included in Fig. 1. Also, a Zipf law is found for the distribution of total liabilities of bankrupted firms in Japan (23, 24).

We obtain that $\zeta_b > \zeta_a$, a discrepancy that could be of potential practical interest. To clarify this point, if A_b is related to A_a by a constant $A_b/A_a \equiv c$, we would observe $\zeta_a = \zeta_b$. However, we observe an increasing relation $A_a/A_b \propto r^{\zeta_b - \zeta_a}$ with rank r, meaning that bankrupt firms with smaller A_a have larger relative adjustments than do bankrupt firms with larger A_b.

Our analysis of bankruptcy probability is, due to data limitation, based on book values. One may argue that a more relevant analysis would be based on market values of assets and liabilities. We now demonstrate that using market instead of book values may in fact lead to similar results. For this purpose, let us consider companies for which we have both market and book value data, namely stocks that comprise the Nasdaq. We begin by finding market capitalization of Nasdaq members for each year from 2002 to 2007. The data are available at Bloomberg L.P. Fig. 2*A* shows the Zipf plot for market capitalization deflated to 2002 dollar values. We find that the market capitalization versus rank for the largest ≈1,000 companies is well described by a Zipf law with exponent $\zeta_M = 1.1 \pm 0.02$, in agreement with ref. 25.

In Fig. 2*B* we repeat the Zipf analysis using, this time, book values of both assets and debt for the same Nasdaq stocks. The scaling exponents we observe in Fig. 2*B* are larger than the exponent observed in Fig. 2*A*. However, market capitalization is best compared with book value of equity $E \equiv A - D$, rather than assets A. In Fig. 2*C*, we find that E also exhibits Zipf scaling with exponent $\zeta_E = 1.02 \pm 0.01$, which is more similar to ζ_M. Therefore, we find qualitatively similar scaling for the existing Nasdaq companies and for companies before they entered into bankruptcy proceedings.

The probability of bankruptcy $P(R)$ is a natural proxy for firm distress (10). Previous studies analyzed defaults of firms traded at NYSE, American Stock Exchange (AMEX), and Nasdaq (10). In contrast, the majority of firms in our dataset are privately held companies. For bankrupt firms in Fig. 3*A* we show $P(R|B)$ for values of the debt-to-assets ratio $0 < R < 4$. We truncate data to avoid outliers as in ref. 11. We find $P(R|B)$ is right-skewed with a maximum at $R \approx 1$, and $\langle R \rangle = 1.4 \pm 1.5$.

Previous studies find that bankruptcy risk of NYSE and AMEX stocks is negatively related to firm size (10). In order to test for firm-size dependence of bankruptcy risk with R as bankruptcy measure, we divide the R values into two subsamples based on their value of A_b. In Fig. 3*A* we demonstrate qualitatively that R is size dependent. The probability density functions (pdfs) for small A_b and large A_b are similar in that they both show peaks at $R \approx 1$. However, firms with smaller assets, as measured by A_b, have a larger probability of high debt-to-assets ratios R than firms with large assets A_b.

In addition, we test for the size dependence by performing the Mann–Whitney U test, which quantifies the difference between the two populations based on the difference between the asset ranks of the two samples. (The null hypothesis is that the distributions are the same.) Because the test statistics U value $= -5.60$, we reject the null hypothesis thus confirming that R depends on A_b at the $p = 0.05$ confidence level.

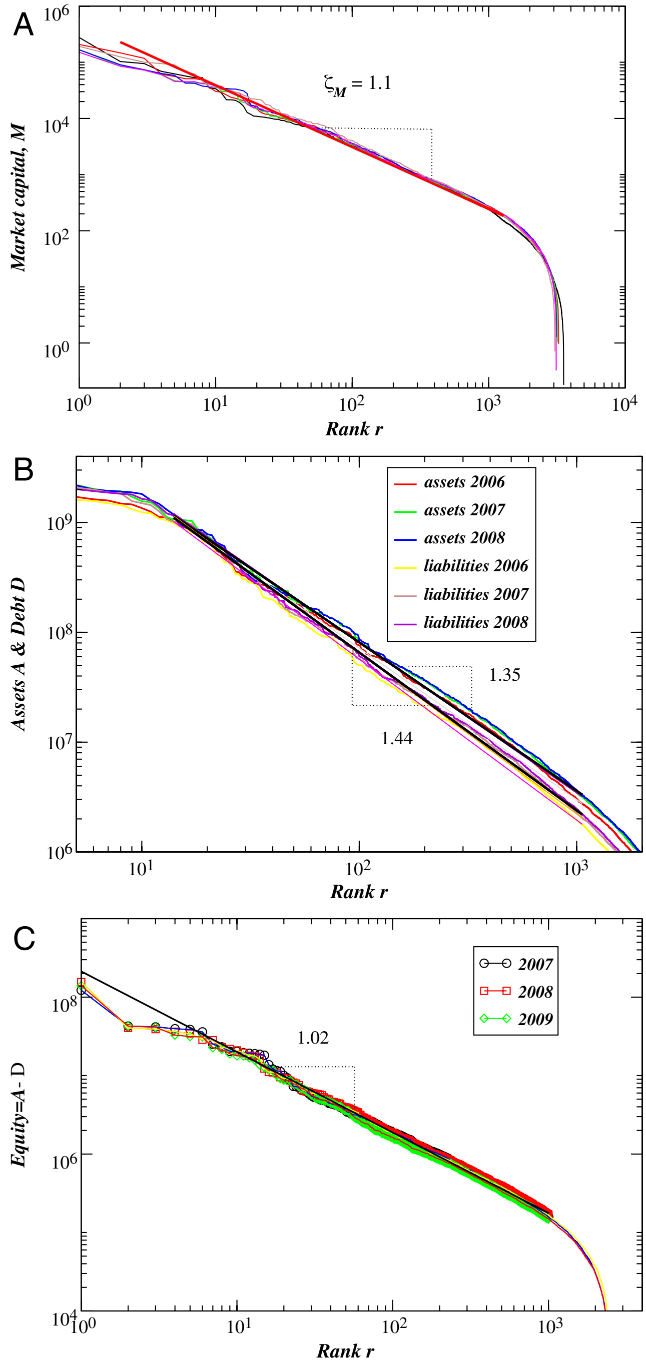

Fig. 2. (*A*) Zipf plot of market capitalization *M* versus rank *r* for the Nasdaq members for each year of 6 y. We find practically the same Zipf law for the largest 1,000 companies as we find for the assets A_a of bankrupted firms in Fig. 1*B*. (*B*) For the Nasdaq firms, both assets and liabilities follow a Zipf plot. (*C*) Book value of equity of stock traded at Nasdaq, defined as assets less liabilities, follows a Zipf law.

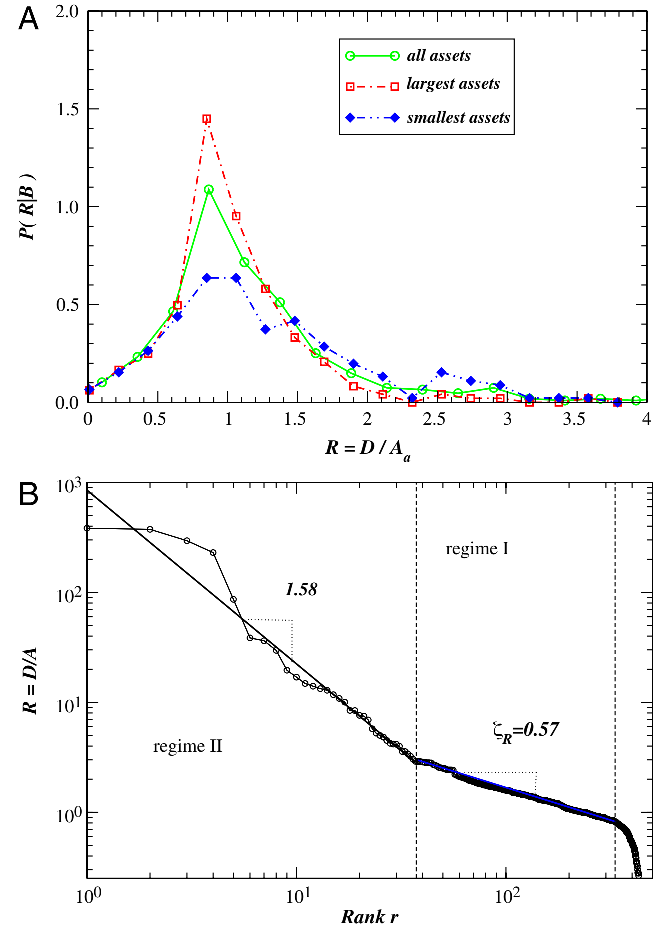

Fig. 3. Bankruptcy risk based on petition book value asset A_b and debt D_b. (*A*) We find the distribution of $R = D_b/A_b$ for the bankrupted firms. One may calculate the probability that a firm with a given ratio *R* will go bankrupt when its ratio is $\leq R$. (*B*) For the ratio values $3 > R > 0.8$ (67% of all data), we show the Zipf plot that can be approximated by a Pareto distribution with $\zeta_R = 0.57$. The same regime we fit with the power-law tail of pdf and obtain $0.79R^{-2.72}$ where the exponent $\zeta' + 1 = 2.73$ is in agreement (see Eq. **2**) with the Zipf exponent $\zeta = 0.57$. For the largest ratio values $R > 3$ (7% of all data), we find a cross-over to a power-law regime with $\zeta_R = 1.58$.

In Fig. 3*B* we analyze the Zipf scaling for large *R*. We find that the Zipf plot can be approximated by two power-law regimes. For ≈300 firms with $0.8 < R < 3$ (regime **I**), we find a power-law regime with $\zeta_R = 0.57 \pm 0.02$. Hence, according to Eq. **2** we conclude that the cumulative distribution of dangerously high *R* values of bankrupt firms decreases faster with $\zeta' \approx 1.72$ for large *R* than the distribution of firm size (20) and firm assets with $\zeta' \approx 1$ (see Fig. 1). For $R > 3$ (7% of all data including predominantly financial firms), we find that the Zipf plot exhibits a significant cross-over behavior to a power-law regime with $\zeta \approx 1.58$.

The conditional probability $P(B|R)$ that an existing firm with debt-to-assets ratio *R* will file for bankruptcy protection may be of significance to rating agencies, creditors, and investors. According to Bayes's theorem, $P(B|R)$ depends on $P(R|B)$ (see Fig. 3), $P(B)$, the probability of bankruptcy for existing firms, and $P(R)$, the probability of an existing company with leverage ratio *R*. In order to estimate $P(R)$, we use the companies constituting the Nasdaq in the 3-y period between 2007 and 2009 as a proxy for existing companies. For this time period, we obtain book value of each firm's assets and liabilities (the latter serving as a proxy for total debt). As a result, we obtain 7,635 *R* values with median value 0.48. For existing Nasdaq members, Fig. 4 shows that the Zipf plot can be approximated by two power-law regimes, where regime **I** with $3.5 > R > 0.9$ yields $\zeta_e = 0.37 \pm 0.01$. Note that regime **I** is similar to the one we find in Fig. 3*B* for bankruptcy data. $P(B)$ may substantially change during economic crises. Interestingly, ref. 26 analyzes the debt-to-GDP (gross domestic product) ratio for countries, in analogy to the debt-to-assets ratio for existing firms, and calculates a Zipf scaling exponent that is approximately the same as the scaling exponent calculated here for existing Nasdaq firms.

We estimate the scaling of $P(B|R)$ using Bayes's theorem,

$$P(B|R) = \frac{P(R|B)P(B)}{P(R)} \approx 0.51P(B)R^{1/\zeta_e - 1/\zeta_R} \approx 0.51P(B)R^{0.95}, \quad [5]$$

where we approximate $P(R|B)$ and $P(R)$ with power laws —$P(R|B) \sim R^{-(1/\zeta_R+1)}\Delta R$ and $P(R) \sim R^{-(1/\zeta_e+1)}\Delta R$. The value of the relevant exponents calculated for regime **I** are as follows: $\zeta_e \approx 0.37$ (see Fig. 4) and $\zeta_R = 0.57$ (see Fig. 3*B*), where $\zeta_R > \zeta_e$ implies that $P(B|R)$ increases with firm indebtedness quantified by R. The prefactor 0.51 calculated for the regime **I** we estimate from the corresponding intercepts in pdfs [see Figs. 3*B* and 4]. In Fig. 4, we find a pronounced cross-over in the Zipf plot for very large values of the R ratio.

In order to test whether market crash and global recession have significant effects on the scaling we find in the bankruptcy data, in Fig. 5 we analyze the Zipf scaling of the large R values for three different 3-y periods. For the period 2004–2006, we find a stable Zipf plot characterized by an exponent $\zeta_R = 0.50 \pm 0.01$ close to the value we found in Fig. 3*B* for all years analyzed. For the period 2001–2003 characterized by the dot-com bubble burst, we find a less pronounced cross-over in the Zipf plot between regime **I** with exponent $\zeta_R = 0.58 \pm 0.01$ and regime **II**. For the period 2007–2009, we find that the Zipf plot exhibits a significant cross-over behavior between regime **I** and regime **II**.

Fig. 5 demonstrates the existence of a relatively stable scaling exponent (between 0.5 and 0.6) in regime **I** over the 9-y period 2001–2009. However, in times of economic crisis, e.g., the period 2007–2009, the exponent in regime **I** increases, implying that firms go bankrupt with larger values of R. According to Eq. **5**, in times of crisis ($\zeta_R \approx 0.6$) $P(B|R) \propto R^{1/\zeta_e - 1/\zeta_R} \propto R^1$ shifts upward compared to times of relative stability ($\zeta_R \approx 0.5$) when $P(B|R) \propto R^{0.7}$. A cross-over in scaling exponents may be useful for understanding asset bubbles.

Model

Our results complement both the literature on default risk as well as the literature on firm growth. According to a study of US firm dynamics, over 65% of the 500 largest US firms in 1982 no longer existed as independent entities by 1996 (27). To explain how firms develop, expand, and then cease to exist, Jovanovic proposed a theory of selection where the key is firm efficiency; efficient firms grow and survive and the inefficient decline and, eventually, fail (15). Many models have been proposed to model default risk (1, 2, 28–31). One strain of that literature (28) develops structural models of credit risk. In these models, risky debt is modeled within an option-pricing framework where an underlying asset is the value of company assets. Bankruptcy occurs endogenously when the value of company assets is insufficient to cover obligations. In contrast, in reduced form models (2) default is modeled exogenously.

In order to reproduce the Zipf law that holds for bankrupt firms, we propose a coupled Simon model, an extension of the Simon model used in the theory of firm growth (13, 14, 16, 17). Here we couple the evolution of both asset growth and debt growth through debt acquisition which depends on a firms assets, and further impose a bankruptcy condition on a firm's assets and debt values at any given time.

Simon Rule for Assets. The economy begins with one firm at the initial time $t \equiv 1$. At each step, a new firm with initial assets $A \equiv 1$ is added to the economy. With a probability p, a new firm i is added to the economy as an individual entity at time t_i. With probability $1-p$, the new firm i is taken over by an already existing firm. The probability that firm i is taken over by an existing firm j is proportional to $A_j(t)$, the number of units in firm j is equal to $(1-p)A_j(t)/\Sigma_k A_k(t)$. Hence, a larger firm is more likely to acquire a firm than a smaller firm. In this expression, the index k runs over all of the existing firms at time t. We use the value $A_j(t)$ to be the proxy for the size of the firm j. Simon found a stationary solution exhibiting power-law scaling, $P(s > x) \propto s^{-\zeta'}$, with exponent $\zeta' = 1/(1-p)$. For an estimate of p, one can investigate venture data to see how venture capitalists dispose of their companies. Even though data suggest $p = 0.5$ (see ref. 32), we use a much smaller value $p = 0.01$ in order to reproduce Zipf plot in Eq. **4**.

Simon Rule for Debt. When a new firm i is created at time t_i, it is assigned debt $D_i(t_i) = m$, where $0 < m < 1$. For simplicity, we use a single m value for all firms. If an existing firm j acquires the new asset $A_i \equiv 1$, then $A_j(t) - A_j(t-1) = 1$, and debt $D_j(t) - D_j(t-1) = m$. Hence, a firm with assets $A_j(t) = N$ has debt $D_j(t) = mN$, implying that the debt-to-assets ratio $R = m$ is the same for all firms.

In order to introduce variation in R ratios across firms, we assume that at each time t_i, a new debt is created in the economy for some company j, so that $D_j(t_i) - D_j(t_i - 1) = 1$. Hence, for each time step, there is a new firm receiving debt $D_i = m$ in addition to

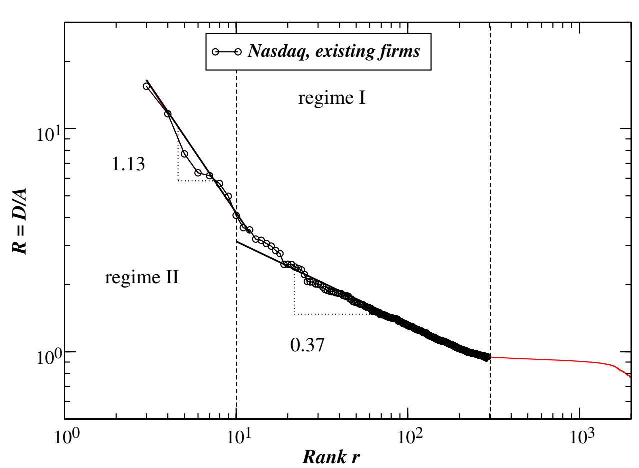

Fig. 4. Zipf plot of debt-to-assets ratio R and rank r for the existing firms of the Nasdaq members over the last 3 y. For the ≈300 ratio values smaller than 3.5 and larger than 0.95 the Zipf plot has exponent 0.37. The same regime we fit with the power-law tail of pdf and obtain $1.54R^{-3.6}$ where the exponent $\zeta' = 3.6$ agrees (see Eq. **2**) with the Zipf exponent $\zeta = 0.37$.

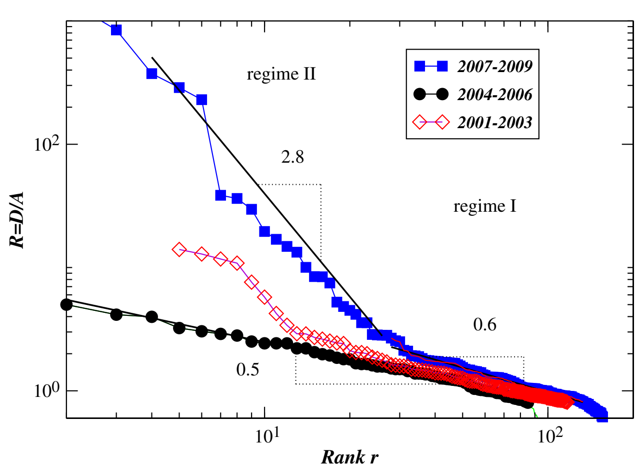

Fig. 5. Zipf plot of debt-to-assets ratio R versus rank r for the bankrupt firms for the three different 3-y subperiods. During the last 3 y characterized by recession, the Zipf plot exhibits a cross-over behavior. A smaller cross-over in the Zipf plot also exists for the period 2001–2003 characterized by the dot-com bubble burst.

firm j receiving one unit of debt, where generally $i \neq j$. The newly created units of debt are acquired with probability proportional to $A_j(t)$. Hence, the Simon laws controlling the growth of debt $D_j(t)$ and the growth of assets $A_j(t)$ are coupled. In our model, richer firms become more indebted, but also acquire new firms with larger probability.

In Fig. 6*A*, we perform the numerical simulation of the model by generating 500,000 Monte Carlo time steps. We calculate the

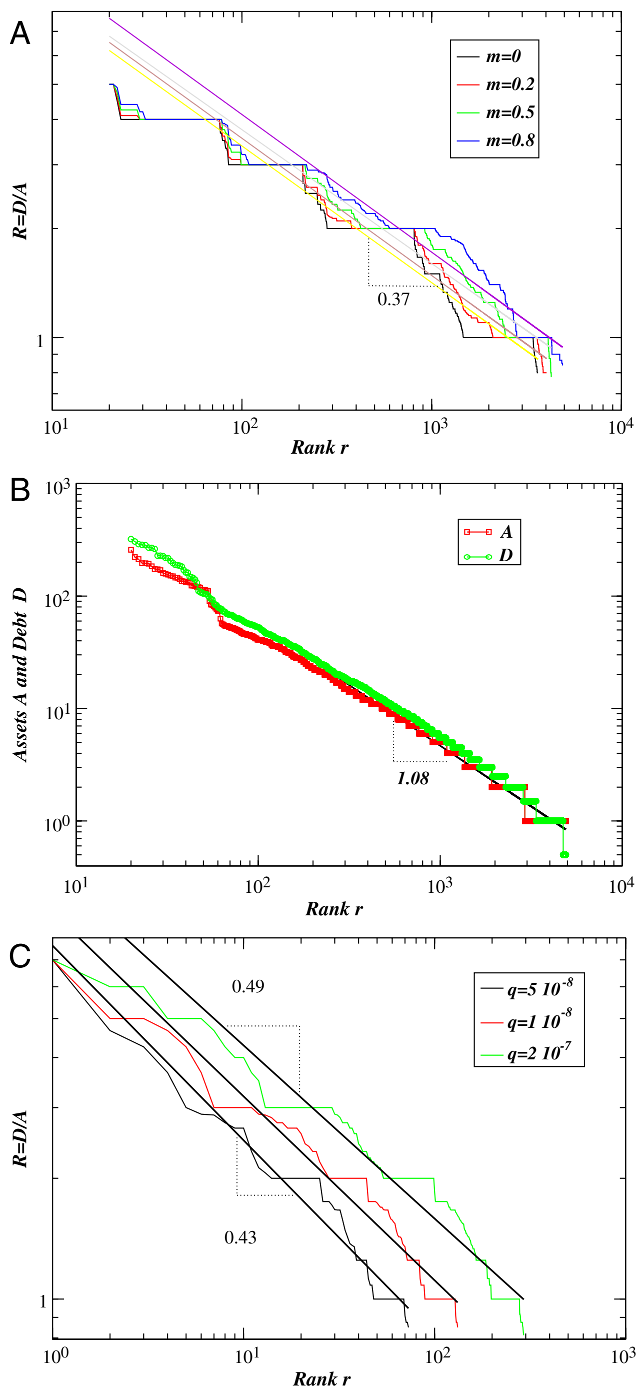

Fig. 6. Model results. (*A*) Zipf plot of debt-to-assets ratio R versus rank r for the firms generated by the model (compare with Fig. 4) when bankruptcy is not included. In order to understand plateaus in the figure, note that both assets and debt in the model exhibit integer values. (*B*) For each asset and debt, the Zipf plot displays a power law $R \sim r^{-\zeta}$. (*C*) Zipf plot of R versus rank as a function of bankruptcy rate parameter q. With decreasing q, the slope increases slightly.

Zipf distribution of the debt-to-assets ratio R for different choices of m. Even though debt and hence R increases with m, the slope of the Zipf plot for R versus rank practically does not depend on the value of m. Unless stated otherwise, in other simulations we set $m = 0.5$.

Following ref. 33, we consider the continuous-time version of our discrete-time model. In this case, $D_j(t)$ and $A_j(t)$ are continuous real-valued functions of time. Further, we assume that the rate at which $D_j(t)$ changes in time is proportional to the assets size $A_j(t)$. Hence, following this assumption, $D_j(t) = (1+m)A_j(t)$ because of the acquisition of additional debt. Therefore, because $A_j(t) = t/t_j$ (33), then $D_j(t) = (1+m)t/t_j$. The cumulative probability that a firm has debt size $D_j(t)$ smaller than D is, therefore, $P[D_j(t) < D] = P[t_j > (1+m)t/D]$. In the Simon model we add new firms at equal time intervals. Thus, each value t_i is realized with a constant probability $P(t_j) = 1/t$. It follows that

$$P[t_j > (1+m)t/D] = 1 - (1+m)t/Dt = 1 - (1+m)/D. \quad [6]$$

Hence, Eq. **6** should be considered as the Zipf law for debt in the case when there is no possibility of bankruptcy (see Eq. **3**).

Firm Bankruptcy. Up to now, debt has been modeled as riskless. We now introduce bankruptcy into the coupled Simon model. We assume that for each firm there is a likelihood of bankruptcy, which depends on the volatile firm asset value (28). In order to be consistent with our empirical findings, we assume that the firm j that was created at time t_j files for bankruptcy with probability $qR^{0.95}$ (see Eq. **6**), where q is the bankruptcy rate parameter, related to $P(B)$ in Eq. **6**. In the hazard model, the hazard rate is the probability of bankruptcy as of time t, conditional upon having survived until time t (11). In our model, once firm j files for bankruptcy, part of its debt is lost (restructured) and the firm starts anew with debt equal to $D_j = mA_j$. We do not assume a merger or a liquidation and a firm's probability of failure does not depend on its age (11). Besides bankruptcy, a firm may leave an industry through merger and voluntary liquidation (9).

Next we perform 500,000 Monte Carlo time steps for the model with the possibility of bankruptcy. Fig. 6*B* presents Zipf distribution for firm asset and debt values for all of the existing firms. Each of these distributions is in agreement with the Zipf law and Eq. **6**. In Fig. 6*C*, for the subset of bankrupt companies, we show the Zipf distribution for R using three different values of the bankruptcy rate q. Note that q is supposed to be small. Namely, with $q = 10^{-7}$ and with 500,000 time steps representing 1 y, $500{,}000q$ represents a probability per year that a company files for bankruptcy during a period of 1 yr, ≈0.05 in our case. Our result for the annual probability of bankruptcy should be compared with the average default rate ≈0.04, calculated in the period 1985–2007 (34). We see that model predictions approximately correspond to the empirical findings.

Our model can be extended in different ways, including mergers between firms. First, although the Simon model assumes that, at each time increment a new unit is added, we can assume that the number of new units grows as a power law t^{θ} (35). By using a continuous-time version of a discrete-time model, we obtain $P[D_j(t) < D] = P[t_j > (1+m)t/D] = 1 - (\frac{1+m}{D})^{1+\theta}$, where we use $P(t_i > t_0) = \int_{t_0}^{t} dt t^{\theta} / \int_0^t dt t^{\theta}$. Second, Jovanovic and Rousseau (32) found that mergers contribute more to firm growth than when a firm takes over a small new entrant. In order to incorporate mergers into the Simon model, we assume that at each time t, a single merger between a pair of firms occurs with probability p', where two firms are randomly chosen. Ref. 36 reported that, in more than two-thirds of all mergers since 1973, the Tobin Q value of the acquisition firm exceeded the Tobin Q value of the target firm, where Q is Tobin's ratio similarly defined as D ratio in Eq. **1**. To this end, we assume that if $A_j > A_i$ when a merger occurs, $A_j = A_j + A_i$ and $A_i = 0$. Thus, the more-rich firm j buys the less-rich

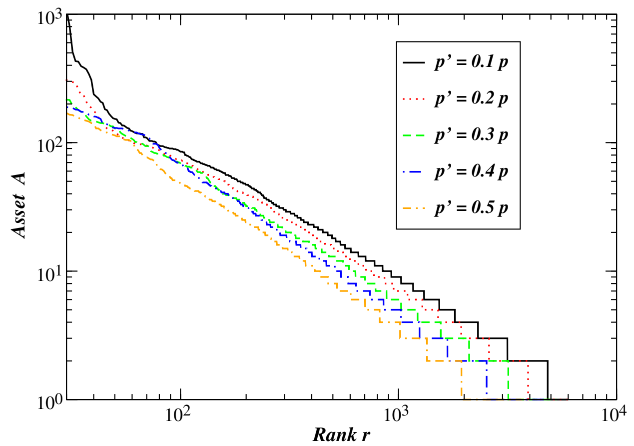

Fig. 7. Scale-free persistence in the Simon model with mergers. Zipf plot of value $A_j(t)$—the value of assets of the firm j—versus rank r for the Simon model with a merger parameter p' representing the probability that a pair of firms will merge. With increasing p', ζ slowly increases.

firm i resulting in the elimination of firm i as an individual entity. In Fig. 7, we show that the inclusion of mergers does not change the scale-free nature of the Simon model. In these simulations, we use a varying merger probability p', and $p = 0.01$ with 1 million time steps. With increasing p', the Zipf exponent ζ slowly decreases. Note that, with 1 million time steps, if $p' = 0.5p$, and with $p = 0.01$, then approximately 5,000 mergers occur.

Before concluding, we note that market capitalization as well as book value of assets, liabilities, and equity for the stocks traded at Nasdaq exhibit Pareto scaling properties. Pareto scaling properties are not trivial consequences of the scaling (20) because, for companies traded at NYSE, we do not find similar power-law scaling for market capitalization (see Fig. 8*A*) and book value of equity. However, the book value of assets and liabilities for NYSE stocks follows a Pareto law with exponents that are slightly larger than those we find for Nasdaq stocks (see Fig. 8*B*). Our results reveal a discrepancy in scaling of market capitalization and book value of equity obtained from different exchange markets (e.g., Nasdaq and NYSE).

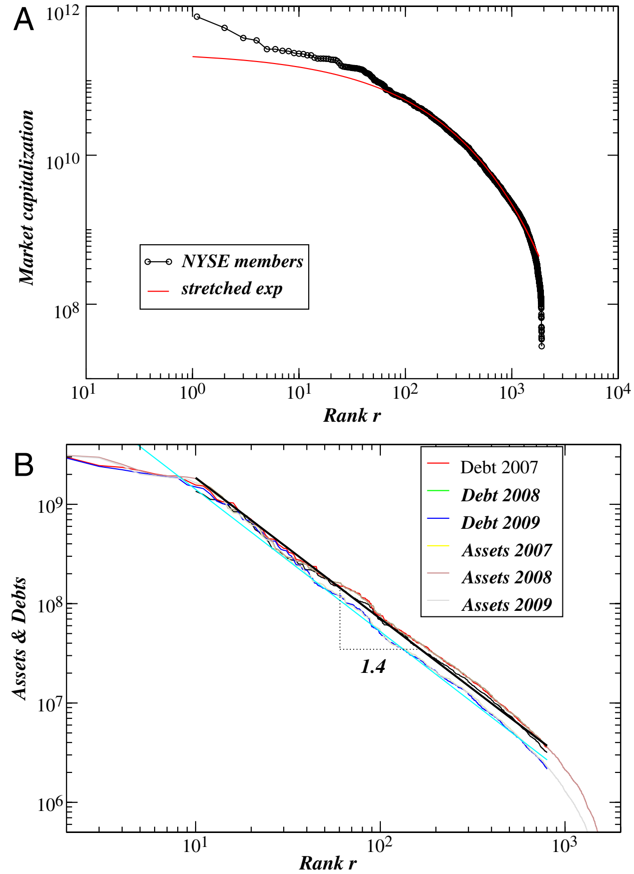

Fig. 8. Zipf plot of (*A*) market capitalization *M* versus rank r and (*B*) assets and debt versus rank r for the NYSE members for year 2007. The curve in *A* follows a stretched exponential $\exp(-r^{\beta}/\tau)$ with $\beta = 0.5$ and $\tau = 45$.

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